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Computation of eigenvalues and solutions of regular Sturm–Liouville problems using Haar wavelets

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Abstract

The paper presents a novel method for the computation of eigenvalues and solutions of Sturm–Liouville eigenvalue problems (SLEPs) using truncated Haar wavelet series. This is an extension of the technique proposed by Hsiao to solve discretized version of variational problems via Haar wavelets. The proposed method aims to cover a wider class of problems, by applying it to historically important and a very useful class of boundary value problems, thereby enhancing its applicability. To demonstrate the effectiveness and efficiency of the method various celebrated Sturm–Liouville problems are analyzed for their eigenvalues and solutions. Also, eigensystems are investigated for their asymptotic and oscillatory behavior. The proposed scheme, unlike the conventional numerical schemes, such as Rayleigh quotient and Rayleigh–Ritz approximation, gives eigenpairs simultaneously and provides upper and lower estimates of the smallest eigenvalue, and it is found to have quadratic convergence with increase in resolution.

Keywords: Haar wavelets; Operational matrix; Lagrange multipliers; Eigenpairs

1. Introduction

Sturm–Liouville eigenvalue problems (SLEPs) are ubiquitous in applied mathematics. In recent years there has been a considerable renewal of interest in the SLEPs, from the point of view of both mathematics and their applications to physics and engineering. For many important applications in science and engineering it is required to determine the eigenvalues as well as the corresponding eigenfunctions. In fact, the general theory of eigenvalues and eigenfunctions is one of the deepest and richest parts of mathematical physics. In applications, for instance, involving vibration and stability of deformable bodies, the vital piece of information required is the smallest eigenvalue [3,7]. Engineers are often interested in the location of the smallest eigenvalue since this gives potentially the most visual structure of dynamic systems. The seismic damage to a structure can be catastrophic if its fundamental frequency (related in some way to the smallest eigenvalue) is of the same order as the frequency of the earthquake [3]. The eigenvalues are also crucial in finding the stability region of solutions of SLEPs [1]. Generally, finding the eigenvalues and the corresponding nontrivial solutions poses a formidable task. The variational formulation of boundary value problems in general has some advantages. The basic idea involved in our method is therefore to reformulate SLEPs into variational

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forms and solve them using properties of Haar wavelets [8]. This strategy is found to work well. Localization and orthogonality properties of Haar wavelets combined with the direct method of the calculus of variations reduce the regular Sturm–Lioville problems to a system of algebraic equations. The most distinguishing feature of this scheme is the useful role played by the Lagrange multiplier, which gives the eigenvalues and eigenvectors at a stretch, reducing the computational cost substantially. The existing methods such as *Rayleigh–Ritz* (R–R) approximation and *Rayleigh quotient* (RQ) do not have this advantage. Moreover, these methods do not work well except in simple cases, because it is too difficult to guess good trial functions and more accuracy requires more trial functions, but more trial functions require more time to compute and more space to store. The benefits of the proposed scheme are: from the distribution pattern of the eigenvalues their asymptotic behavior can be predicted; secondly, one can estimate and hence locate the smallest eigenvalue by computing its bounds; thirdly, useful information regarding oscillatory behavior of eigenfunctions can be obtained, whereas, the classical RQ is of limited practical value, since it gives only an upper estimate of the smallest eigenvalue.

In this paper we discuss few simple applications of the variational approach to SLEPs. Recently, Haar wavelets have been used in the solution of variational problems in [8]. The method developed by Hsiao applies only to linear and nonlinear variational problems. We extend the method of Hsiao for solving SLEPs and thereby increase the range of applicability of Haar wavelets. The motivation for using Haar wavelets stems from their simplicity, orthonormality, computational convenience and rapidity of convergence in approximating functions and operators.

Over the last couple of decades, wavelets in general have gained a respectable status due to their applications in various disciplines and as such have many success stories. Notable impacts of their studies are in the fields of signal and image processing, numerical analysis, differential and integral equations, tomography, etc. One of the most successful applications of wavelets has been in image processing. The FBI has adopted a wavelet-based compression algorithm (leading to wavelet-chips) for fingerprint compression. Wavelets have the ability to represent functions at different levels of resolution, which allows developing a hierarchy of approximate solutions of equations. Compactly supported wavelets are localized in space, wherein solutions can be refined in regions of sharp variations/transients without going for new grid generation, which is the common strategy in classical numerical schemes.

Mathematical modeling of many problems of interest (in general) can be expressed in variational formulations. This class of problems has attracted the attention of the best brains in mathematics, which are classical in nature, to be found in many good textbooks [3,7]. They have been of interest and utility in the accurate representation of physical phenomena and in their analysis. Sturm–Liouville problems, many of which have their origin in the calculus of variations, are of paramount importance in science and engineering. Under suitable conditions these system can be reduced to standard forms and solved accurately using conventional procedures in favorable cases. The method of orthogonal expansion is one of the useful methods in this category. The key to a viable alternative is the choice of a suitable basis. Recent studies have shown that, Haar wavelets are the right choice for this purpose, being the simplest of all the wavelets. Hsiao has demonstrated their superiority over other orthogonal bases such as Walsh and Fourier [4,8]. While they have drawbacks, chiefly lack of continuity, they still illustrate in the most direct way some of the main features of wavelet decomposition and reconstruction. For this reason we shall consider in some detail the properties that make them suitable for numerical applications and to a greater advantage in achieving tradeoff between speed and accuracy. Hsiao [8] has successfully utilized Haar wavelets in his work on optimization theory, control theory, stiff differential equations, etc., wherein the efficiency of the method is demonstrated convincingly. The rest of the paper is organized as follows. Section 2 contains the necessary theoretical background (mathematical basis) and tools for the solution of Sturm–Liouville problems. In Section 3 we discuss variational formulation of SLEP. In Section 4 we present the method of solution and computational aspects of implementation. Section 5 deals with various typical Sturm–Liouville problems amenable to the Haar wavelet series method (HWSM) and their solutions. Section 6 contains summary of the results obtained and their accuracies compared with other available solutions. We conclude with discussion on the results obtained and scope for future work.

2. Some properties of Haar wavelets and the associated matrices

In this section we briefly summarize the properties of Haar wavelets and develop some computational strategy to establish notation and terminology.

It is useful to have alternative notations for Haar functions $\varphi_{n.k}$ and $\psi_{n,k}$.

We define $h_0(x) = 1$, $x \in [0, 1)$ and

$$h_1(x) = \begin{cases} 1, & x \in [0, \frac{1}{2}), \\ -1, & x \in [-\frac{1}{2}, 1), \\ 0, & x \notin [0, 1). \end{cases}$$

The general Haar function is defined by

$$h_i(x) = h_1(2^j x - k), \quad i = 2^j + k, \quad j \geqslant 0, \quad 0 \leqslant k < 2^j.$$

As Haar wavelets form an orthonormal basis [12] for $L_2([0, 1))$ any $y(x) \in L_2([0, 1))$ can be expressed in the form

$$y_(x) = \sum_{i=0}^{\infty} c_i h_i(x), \quad i \in \{0\} \cup N, \tag{2.1}$$

where the Haar coefficients c_i are given by

$$c_0 = \int_0^1 y(x) h_0(x)\,\mathrm{d}x,$$

$$c_i = 2^j \int_0^1 y(x) h_i(x)\,\mathrm{d}x. \tag{2.2}$$

The coefficients c_i are chosen so as to achieve "mean convergence" in (2.1). As a consequence, the so-called "least squares" error

$$\in = \int_0^1 \left[y(x) - \sum_{i=0}^{m-1} c_i h_i(x) \right]^2 \mathrm{d}x, \quad m = 2^j, \quad j \in \{0\} \cup N$$

is minimized.

Therefore, the series (2.1) converges in the mean square sense.

In practice, only the first m terms are considered, where m is an integral power of 2. Then from (2.1), we get

$$\begin{aligned} y(x) &\approx \sum_{i=0}^{m-1} c_i h_i(x) = \mathbf{c}_{(m)}^{\mathrm{T}} \mathbf{h}_{(m)}(x), \\ \mathbf{c}_{(m)} &= [c_0, c_1, c_2, \ldots, c_{m-1}]^{\mathrm{T}}, \\ \mathbf{h}_{(m)}(x) &= [h_0(x), h_1(x), \ldots, h_{m-1}(x)]^{\mathrm{T}}, \end{aligned} \tag{2.3}$$

where T stands for transpose. The integration of Haar wavelets is expandable into a Haar series with coefficient matrix P, defined by

$$\int_0^x h_m(\tau)\,\mathrm{d}\tau \approx P_{(m\times m)} h_m(x) \quad x \in [0, 1), \tag{2.4}$$

also called the operational matrix of integration which satisfies the following recursive formula with $P_{(1\times 1)} = \frac{1}{2}$ and $m = 2^j$, $m > 2$ [9]:

$$P_{(m\times m)} = \frac{1}{(2m)} \begin{bmatrix} 2mP_{(\frac{m}{2}\times\frac{m}{2})} & -H_{(\frac{m}{2}\times\frac{m}{2})} \\ H^{-1}_{(\frac{m}{2}\times\frac{m}{2})} & 0_{(\frac{m}{2}\times\frac{m}{2})} \end{bmatrix}, \tag{2.5}$$

where

$$H_{(m\times m)}=[h_{(m)}(x_0),h_{(m)}(x_1),\ldots,h_{(m)}(x_{m-1})],$$

$$\frac{i}{m}\leqslant x_i<\frac{(i+1)}{m}\quad\text{and}\quad H^{-1}_{(m\times m)}=\left(\frac{1}{m}\right)H^{\mathrm{T}}_{(m\times m)}\,\mathrm{diag}(r),$$

$$r=\left[1,1,2,2,4,4,4,4,\ldots,\underbrace{\frac{m}{2},\frac{m}{2},\ldots,\frac{m}{2}}_{\frac{m}{2}\text{elements}}\right]^{\mathrm{T}},\quad m>2.$$

In the study of variational problems via Haar wavelets, it is required to evaluate the integration of $h_m(x)h_m^{\mathrm{T}}(x)$, which in matrix form is

$$\int_0^1 h_{(m)}(\tau)h^{\mathrm{T}}_{(m)}(\tau)\,\mathrm{d}\tau=\begin{bmatrix} I_{(2\times 2)} & & & 0\\ & \frac{1}{2}I_{(2\times 2)} & & \\ & & \frac{1}{4}I_{(4\times 4)} & \\ 0 & & & \frac{2}{m}I_{(\frac{m}{2}\times\frac{m}{2})}\end{bmatrix}=K_{m\times m}\quad\text{for } m>2, \tag{2.6}$$

where $I_{2k\times 2k}$ are unit matrices.

3. Theoretical background

We define the Sturm–Liouville [3] operator on a bounded interval $[a,b]$ by

$$Ly=-(py')'+qy,\ \text{ where }\ p\in C^1[a,b],\ q\in C[a,b],\ p>0,\ q\geqslant 0.$$

We are interested in the SLEP

$$Ly=\lambda ry,\quad \lambda\in R,\ \ r\in C[a,b]\ \text{ and }\ r>0, \tag{3.1}$$

subject to the boundary conditions in the form

$$\left.\begin{array}{l}\alpha y(a)+\beta y'(b)=0\\ \alpha' y(a)+\beta' y'(b)=0\end{array}\right\}. \tag{3.2}$$

Values of λ for which (3.1), (3.2) have a nontrivial solution are called eigenvalues and nontrivial solutions y corresponding to λ are called eigenfunctions. The pair (λ, y) is called an eigenpair for the SLEP (3.1), (3.2).

Using standard variational methods, we look for a minimizer of the functional

$$J(y)=\int_a^b [py'^2+qy^2]\,\mathrm{d}x, \tag{3.3}$$

which corresponds to the Euler–Lagrange equation given by the Sturm–Liouville equation, over $\{y\in H_0^1([a,b]):\int_a^b y^2\,\mathrm{d}x=1\}$, where H_0^1 is the Sobolev space, i.e.,

$$H_0^1=\{f/f,\ f'\in L_2[a,b],\ f(a)=f(b)=0\},$$

so that a minimizer will yield the equation

$$J'(y)=\lambda y. \tag{3.4}$$

The condition $\int_a^b y^2\,\mathrm{d}x=1$ is referred to as isoperimetric or normalization condition.

Computation of eigenvalues using classical techniques:

The RQ, which is used to approximate the smallest eigenvalue, is given by $\lambda_1 = \|\nabla\omega\|^2/\|\omega\|^2$, ω being any trial function and ∇ the gradient operator. The method of (R–R) approximations uses the determinant as a tool to approximate the first few eigenvalues as shown below [11].

If $\omega_1, \omega_2, \ldots, \omega_n$ are trial functions, then we define $A = (a_{ij})$, $B = (b_{ij})$,

where $a_{ij} = \int_0^1 \nabla\omega_i \nabla\omega_j \,\mathrm{d}x$ and $b_{ij} = \int_0^1 \omega_i\omega_j \,\mathrm{d}x$.

The numerical approximations to the first n eigenvalues are given by the roots of $|A - \lambda B| = 0$.

The computed results are given in Tables 4–6.

4. Method of solution

We first recast problem (3.1) and (3.2) as a variational problem. We solve this problem via Haar wavelets for $a = 0$, $b = 1$, as Haar functions are defined in [0, 1). Here, for simplicity of our presentation, we further limit ourselves to $\alpha = \alpha' = 1$, $\beta = \beta' = 0$. The formulation for the general boundary conditions can be done with slight modifications.

We write $J(y) = \int_0^1 (py'^2 + qy^2)\,\mathrm{d}x$ and consider the problem of finding extremals for J subject to boundary conditions $y(0) = y(1) = 0$ and the isoperimetric constraint $I(y) = \int_0^1 y^2\,\mathrm{d}x = 1$. This is equivalent to finding extremals of $J^* = J - \lambda(\int_0^1 y^2\,\mathrm{d}x - 1)$, λ being the Lagrange's multiplier [3].

Assuming $y'(x) = \sum_0^{m-1} c_i h_i$ and expressing all functions involved in terms of Haar functions and using the results of Section 2, we obtain $J^* = J^*(c_0, c_1, \ldots, c_{m-1})$.

Applying the necessary condition for extremization and solving for c_i, $i = 0, 1, \ldots, m-1$ the desired solution can be obtained. Interestingly, the Lagrange multiplier plays the role of an eigenparameter and the isoperimetric condition simply scales the eigenfunctions.

5. Numerical examples

We consider three SLEPs taken from [3], two of which have polynomial coefficients and the third one has periodic coefficients.

Example 1. We first consider *Halm's equation*

$$\begin{aligned}&(1+x^2)^2 y''(x) + \lambda y(x) = 0,\\ &y(0) = y(\pi) = 0.\end{aligned} \tag{5.1}$$

Normalizing the interval $[0, \pi]$ by using $x = \pi t$, Eq. (5.1) is transformed into

$$\begin{aligned}&\frac{(1+\pi^2 t^2)^2}{\pi^2} y''(t) + \lambda y(t) = 0,\\ &y(0) = y(1) = 0.\end{aligned} \tag{5.2}$$

It has the variational form

$$J(y) = \int_0^1 \left[\frac{(1+\pi^2 t^2)}{\pi^2} y'^2 - \lambda(y^2 - 1)\right] \mathrm{d}t, \tag{5.3}$$

under boundary conditions (5.2). Using Eqs. (3.3)–(3.6) in (5.3) we get

$$J(y) = \int_0^1 [\mathbf{d}_m^{\mathrm{T}} \mathbf{h}_m(t) \mathbf{h}_m^{\mathrm{T}}(t) \mathbf{d}_m \mathbf{c}_m^{\mathrm{T}} \mathbf{h}_m(t) \mathbf{h}_m^{\mathrm{T}}(t) \mathbf{c}_m - \lambda \mathbf{c}_m^{\mathrm{T}} P_{m\times m} \mathbf{h}_m(t) \mathbf{h}_m^{\mathrm{T}}(t) P_{m\times m}^{\mathrm{T}} \mathbf{c}_m + \lambda]\,\mathrm{d}t, \tag{5.4}$$

where $(1+\pi^2 t^2)/\pi^2 \approx \mathbf{d}_m^{\mathrm{T}} \mathbf{h}_m(t)$.

$$\text{Let } y'(t) \approx \sum_{i=0}^{m-1} c_i h_i = \mathbf{c}_m^{\mathrm{T}} \mathbf{h}_m. \tag{5.5}$$

Table 1
Comparison of computed values of the first eigenvalue λ_1 and also solutions for different t of Example 1 using Haar wavelet series and finite difference methods, when $m = 8$

t	HSWM	FDM@
0	0	0
0.125	0.951799	0.971055
0.250	1.019591	1.262562
0.375	1.332162	1.432585
0.5	1.14197	1.240932
0.625	1.238912	0.963715
0.75	0.653216	0.652278
0.875	0.327256	0.328067
1	0	0

$\lambda_1 = 6.0401$ (HWSM), 5.2208 (FDM).

Table 2
Comparison of computed values of the first eigenvalue λ_1 and solutions of Example 1 using Haar wavelet series and finite difference methods, when $m = 16$, for different t

t	HSWM	FDM@
0	0	0
0.0625	0.53478	0.53476
0.125	0.97108	0.97105
0.1875	1.12278	1.12256
0.25	1.41567	1.41585
0.3125	1.46169	1.46156
0.375	1.43288	1.43275
0.4375	1.35224	1.35256
0.5	1.24077	1.24093
0.5625	1.10789	1.10887
0.625	0.96355	0.96371
0.6875	0.81076	0.81050
0.75	0.65298	0.65227
0.8125	0.49145	0.49103
0.875	0.32856	0.32806
0.9375	0.16550	0.16420
1	0	0

$\lambda_1 = 5.42326$ (HWSM), 5.09980 (FDM).

Integrating (5.5) from 0 to t and using (3.4) together with $y(0) = 0$, we get

$$y(t) \approx \mathbf{c}_m^{\mathrm{T}} P_{m\times m}\mathbf{h}_m(t). \tag{5.6}$$

Eq. (5.6) and the boundary condition $y(1) = 0$ imply $c_0 = 0$.

Hence,

$$J(c_1, c_2, \ldots, c_m) = d_m^{\mathrm{T}} K_{m\times m}\mathbf{d}_m\mathbf{c}_m^{\mathrm{T}} K_{m\times m}\mathbf{c}_m - \lambda c_m^{\mathrm{T}} P_{m\times m} K_{m\times m} P_{m\times m}\mathbf{c}_m + \lambda, \tag{5.7}$$

i.e., when $m = 8$, the number of terms in the Haar wavelet series in (5.7) are

$$\begin{aligned} J = {} & 0.05066c_3^2 + 0.02533c_4^2 + 0.02533c_5^2 + 0.02533c_6^2 + 0.02533c_7^2 \\ & + (0.10132 - 0.13924)c_1^2 + (0.05066 - 0.01657\lambda)c_2^2 + 1.69744\lambda - 0.01657c_3^2\lambda \\ & - 0.00165c_4^2\lambda - 0.00165c_5^2\lambda - 0.00663c_3c_6\lambda - 0.00165c_6^2\lambda + c_1(-0.05304c_2 - 0.05304c_3 \\ & - 0.00663c_4 - 0.01989c_5 - 0.01989c_6 - 0.00630c_7)\lambda - 0.00630c_3c_7\lambda - 0.00165c_7^2\lambda \\ & + c_2(-0.00663c_4\lambda - 0.00663c_5\lambda). \end{aligned}$$

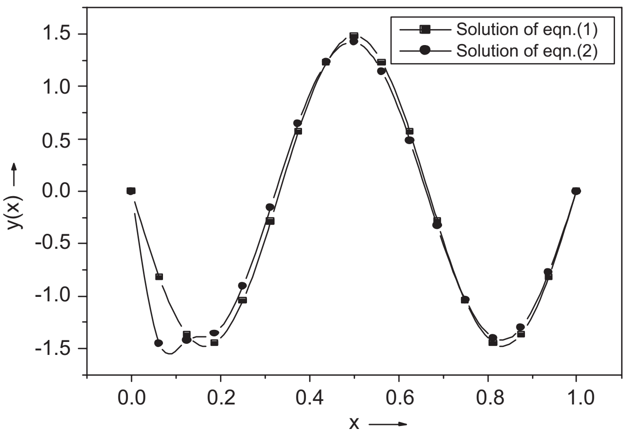

Fig. 1. Verification of Sturm comparison theorem (9).

Table 3
Comparison of computed value of the first eigenvalue λ_1 and also solutions of Example 2 using Haar wavelet series and finite difference methods, when $m = 8$, $n = 0$

x	HWSM	FDM	Exact@
0	0	0	0
0.125	0.55185	0.57402	0.54119
0.25	1.01959	1.06066	1
0.375	1.33217	1.38582	1.30656
0.5	1.44191	1.49291	1.41421
0.625	1.33217	1.38582	1.30656
0.75	1.01959	1.06066	1
0.875	0.55185	0.57402	0.54119
1	0	0	0

$\lambda_1 = 11.1289$ (HWSM), 10.7434 (FDM), 10.8696 (Exact).

We solve $\frac{\partial J}{\partial \mathbf{c}} = 0$, $\frac{\partial J}{\partial \lambda} = 0$, to get

$$y'(t) = (0, 2.88384, 1.19453, 1.19453, 0.33605, 0.81123, 0.81123, 0.33602)\mathbf{h}_8^{\mathrm{T}}$$

and $\lambda = 6.0460$. Eq. (5.6) gives the Haar wavelet series solution corresponding to the smallest eigenvalue.

Next, we seek a numerical solution of (5.2) using the finite difference method (FDM) as an accuracy check. We consider a grid with $h = \frac{1}{8}.t_j = jh$, $j = 1, 2, \ldots, 7$.

The finite difference form of Eq. (5.2) is

$$\frac{(1 + \pi^2 t_i^2)}{\pi^2} \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \lambda y_i = 0 \tag{5.8}$$

and boundary conditions (5.2) imply $y_0 = y_8 = 0$.

If $Ay = \lambda y$ represents the matrix formulation of (5.8), then

$$A = \begin{pmatrix} -2.664 + 0.146\lambda & 1.3322 & 0 & 0 & 0 & 0 & 0 \\ 2.6142 & -5.228 + 10.146\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 5.70213 & -11.404 + 0.146\lambda & 5.70213 & 0 & 0 & 0 \\ 0 & 0 & 12.0229 & -24.045 + 0.146\lambda & 12.0229 & 0 & 0 \\ 0 & 0 & 0 & 23.5741 & -47.148 + 0.146\lambda & & 0 \\ 0 & 0 & 0 & 0 & 42.9242 & -85.848 + 0.146\lambda & 42.924 \\ 0 & 0 & 0 & 0 & 0 & 73.2123 & -146.425 + 0.146\lambda \end{pmatrix}.$$

Table 4
Comparison of the first eigenvalue and solutions of Example 2 using Haar wavelet series and finite difference methods, when $m = 16$, $n = 0$

x	HWSM	FDM	Exact@
0	0	0	0
0.0625	0.27521	0.275899	0.275899
0.125	0.54181	0.541196	0.541196
0.1875	0.78549	0.785695	0.785695
0.25	1.00482	1	1
0.3125	1.17851	1.17588	1.17588
0.375	1.31285	1.30656	1.30656
0.4375	1.38376	1.38704	1.38704
0.5	1.41103	1.41421	1.41421
0.5625	1.38376	1.38704	1.38704
0.625	1.31285	1.30656	1.30656
0.6875	1.17851	1.17588	1.17588
0.75	1.00482	1	1
0.8125	0.78549	0.785695	0.785695
0.875	0.54181	0.541196	0.541191
0.9375	0.27521	0.275899	0.275899
1	0	0	0

$\lambda_1 = 10.9334$ (HWSM), 10.8379 (FDM), 10.8696 (Exact) 10 (RQ or R–R) using trial functions $x - x^2$, $x^2 - x^3$.

Table 5
Comparison of the first eigenvalues and solutions of Example 2 using Haar wavelet series and finite difference methods, when $m = 16$, $n = 1$

x	HSWM	FDM@
0	0	0
0.0625	0.31356	0.31345
0.125	0.54445	0.54483
0.1875	0.78972	0.78916
0.25	1.00483	1.00425
0.3125	1.18112	1.18172
0.375	1.31217	1.31285
0.4375	1.39283	1.39181
0.5	1.41991	1.41923
0.5625	1.39256	1.39226
0.625	1.31162	1.31143
0.6875	1.18045	1.18006
0.75	1.00381	0.99082
0.8125	0.78862	0.78763
0.875	0.66599	0.66549
0.9375	0.15425	0.15422
1	0	0

$\lambda_1 = 10.7017$ (HWSM), 10.1195 (FDM), 10.5 (RQ) using trial function $x - x^2$.

The eigenvalues are obtained by solving $|A| = 0$. The lowest eigenvalue is 5.22008. The eigenvector corresponding to this eigenvalue is

$$\{1.06798, 1.51371, 1.50922, 1.30073, 1.00749, 0.68274, 0.34235\}.$$

Table 1 gives the approximate values of $y(t)$ and the smallest eigenvalue of SLEP when $m = 8$, using the HWSM and FDM solution.

Table 2 gives the approximate values of $y(t)$ and the smallest eigenvalue of SLEP when $m = 16$, using the HWSM and FDM solution.

Table 6
Comparison of the first eigenvalues and solutions of Example 2 using Haar wavelet series and finite difference methods, when $m = 16$, $n = 2$

x	HSWM	FDM@
0	0	0
0.0625	0.27721	0.27756
0.125	0.54381	0.54434
0.1875	0.78949	0.78996
0.25	1.00485	1.00488
0.3125	1.18153	1.18075
0.375	1.31286	1.31082
0.4375	1.39372	1.38996
0.5	1.42102	1.41527
0.5625	1.39371	1.38591
0.625	1.31285	1.30323
0.6875	1.18154	1.18066
0.75	1.0048	0.99361
0.8125	0.77949	0.77917
0.875	0.53481	0.53577
0.9375	0.27726	0.27277
1	0	0

$\lambda_1 = 10.3452$ (HWSM), 9.95067 (FDM), 10.1786 (RQ) using trial function $x - x^2$.

Example 2. We next consider the *Titchmarch equation*

$$\begin{aligned} &y''(x) + (\lambda - x^{2n})y(x) = 0, \\ &y(0) = y(1) = 0, \end{aligned} \tag{5.9}$$

where n is a nonnegative integer.

We estimate the least eigenvalue and show that λ_1 satisfies $\pi^2 < \lambda_1 < 11$.
For a lower bound of λ_1 we solve the *Comparison equation* [3]

$$y''(x) + \lambda y(x) = 0.$$

We follow the procedure outlined in Example 1 and obtain the numerical solutions taking $n = 0, 1, 2$. The accuracy of the method is tested by comparing with the exact solution which exists when $n = 0$ and FDM solution when $n = 1, 2$. We verify graphically the implications of the Sturm comparison theorem [10] (as shown in Fig. 1) that the solutions of the Titchmarch equation oscillate more rapidly with increasing n, a remarkable property of eigenfunctions.

Tables 3–6 give computed eigenvalues and solution $y(t)$ of SLEP (5.9) using HWSM and FDM for $m = 8$, 16 and $n = 0, 1, 2$, the integer parameter in (5.9).

The first eigenvalue of the *Comparison equation* $y''(x) + \lambda y(x) = 0$ under Dirichlet's boundary conditions is 9.7434. Combining this with results obtained, we arrive at a sharp estimation of bounds of the first eigenvalue viz $\pi^2 < \lambda_1 < 11$.

Example 3. We consider *Mathieu's equation* [3,5], a more involved and illuminating two-parameter SLEP

$$y'' + (\lambda - 2\theta\cos(2x))y = 0, \tag{5.10}$$

subject to boundary conditions $y(0) = y(\pi) = 0$.

It must be noted that if $\theta \neq 0$, the nontrivial solutions cannot be expressed in closed form. The estimation of the eigenvalues for this problem is more complicated compared to the problems discussed above. We obtain eigenpairs corresponding to a fixed value of $\theta = 5$, demonstrating the fact that the first eigenvalue can even be negative, a distinguishing feature of Mathieu's equation. The results obtained are given in Tables 7 and 8. We also demonstrate graphically the fact that the first eigenfunction has no zeros in (0, 1) and the nth eigenfunction has $n - 1$ zeros in (0, 1) [2,6] (see Fig. 2). For selected values of n and parameter θ, the spectrum of Example 3 is given in Table 8(a). Shifting

Table 7
Comparison of the first eigenvalues and solutions of Example 3 using Haar wavelet series and finite difference methods, when $m = 8$, $\theta = 5$

t	HSWM	FDM@
0	0	0
0.125	−0.99645	−0.99589
0.25	−0.97652	−0.97578
0.375	−0.98608	−0.98592
0.5	−0.99725	−0.99689
0.625	−1.00592	−1.00488
0.75	−1.01389	−1.01275
0.875	−1.02256	−1.02185
1	0	0

$\lambda_1 = -8.06987$ (HWSM), -6.51126 (FDM).

Table 8
Comparison of the first eigenvalues and solutions of Example 3 using Haar wavelet series and finite difference methods, when $m = 16$, $\theta = 5$. (a) Spectrum of eigenvalues of Mathieu's equation for selected values of n and the parameter θ. (b) Comparison of higher eigenvalues for Mathieu's equation obtained from HWSM and FDM corresponding to $\theta = 5$

t	HSWM	FDM
0	0	0
0.0625	−0.96396	−0.96456
0.125	−0.96893	−0.96658
0.1875	−0.97383	−0.97275
0.25	−0.97894	−0.97658
0.3125	−0.98340	−0.98458
0.375	−0.98808	−0.98678
0.4375	−0.99269	−0.99356
0.5	−0.99723	−0.99681
0.5625	−1.00175	−1.00231
0.625	−1.00609	−1.00589
0.6875	−1.01042	−1.01136
0.75	−1.01467	−1.01386
0.8125	−1.01886	−1.01768
0.875	−1.02297	−1.02278
0.9375	−1.02702	−1.02685
1	0	0

(a)

θ	0	0.1	0.4	0.6	0.8	1	2	5
λ	9.8696	1.21105	1.76538	2.13493	2.50488	2.87356	4.72179	−5.7311

(b)

n	n^2	λ_n (FDM)	λ_n (HWSM)
1	1	−5.7311	−5.4665
2	4	2.0992	2.6161
3	9	9.2365	9.4227
4	16	16.648	16.3707
5	25	25.511	24.1471
6	36	36.359	36.6577

$\lambda_1 = -5.46653$ (HWSM), -5.73115 (FDM).

of symmetry of solutions for selected values of the parameter is displayed in Fig. 3. Table 8(b) predicts the asymptotic behavior of higher eigenvalues of Mathieu's equation and these eigenvalues are $\lambda_n = n^2 + \mathrm{O}(1)$, which is consistent with the classical theorem on asymptoticity of the eigenvalues $\lim_{n\to\infty} \lambda_n^{1/2}/n = 1$ [3].

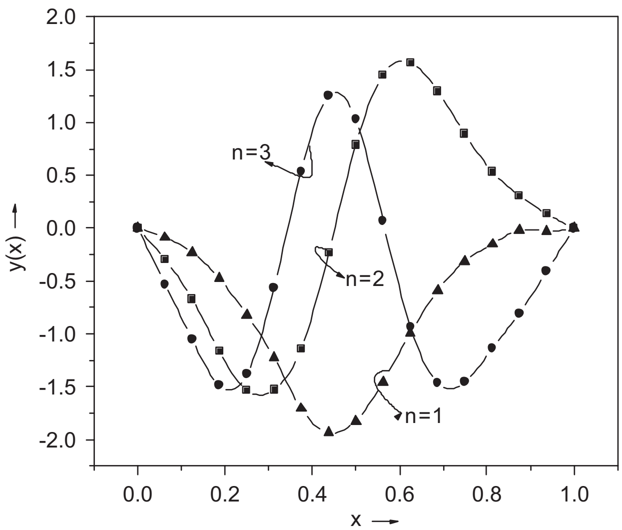

Fig. 2. Higher eigenfunctions of Mathieu's equation for a fixed parameter $\theta = 5$.

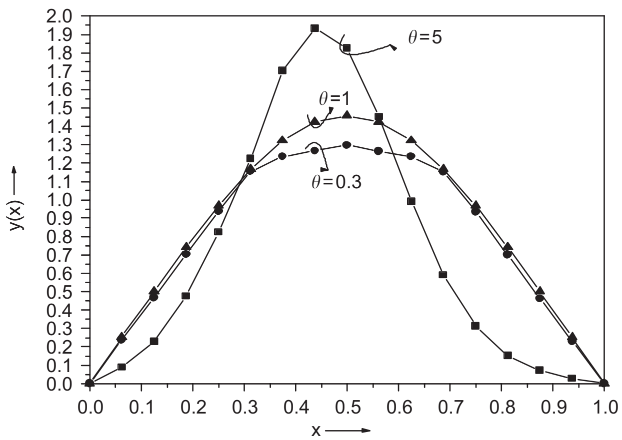

Fig. 3. Solutions of Mathieu's equation for different parameters θ.

6. Discussion and conclusions

To demonstrate the computational efficiency and robustness of the method we have analyzed the solutions of rather difficult SLEPs. Comparison of the results obtained with analytical/numerical solutions confirm that the method proposed is considerably accurate, computationally convenient, consumes less computer time and requires less memory space. The results agree reasonably well with those of solutions obtained from FDM. We can also use the method to narrow the range of the lowest eigenvalue by increasing the grid points, which helps in locating the first eigenvalue accurately. We have demonstrated the usefulness of the technique by showing that convergence to the exact solution, whenever available, is quadratic with increasing resolution. In other words, HWSM is twice as fast as FDM. To that extent, the present method is more efficient than FDM. Moreover, we are able to overcome the limitations of RQ with the proposed scheme. In fact, an advantage of HWSM is not only to provide both upper and lower bounds, but it does not involve the calculation of any integral. We have verified graphically and numerically, some theoretical results from the classical theory regarding oscillatory behavior of eigenfunctions and also the asymptotic behavior of eigenvalues. In view of the properties listed in Section 3, it is reasonable to expect that all eigenvalues are positive. But, contrary to this, two-parameter families of SLEPs do have the first eigenvalue negative. To establish our claim, we have produced numerical evidence, i.e., the value of the parameter giving a negative eigenvalue and indicating the influence of the parameter on the eigenvalues. On the whole, the key advantage of the numerical approach is that eigenpairs of a two-parameter SLEP can be obtained with little additional computational effort once the operational and product matrices are computed and stored and the same can be used for subsequent computations. It may be noted that the procedure described is amenable to generalizations, for instance, SLEPs with mixed boundary conditions. There is also a scope for future work: to cope with singular Sturm–Liouville problems.

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